

FIFTH EDITION



# TRIGONOMETRY



Mark **Dugopolski**



# Get the Most Out of MyLab Math

MyLab™ Math is the teaching and learning platform that empowers instructors to reach every student. By combining trusted author content with digital tools and a flexible platform, MyLab Math personalizes the learning experience and improves results for each student.

- **Flexible platform**—Your course is unique. Whether you’d like to build your own assignments, teach many sections, or set prerequisites, you have the flexibility to easily create *your* course to fit *your* needs.
- **Personalized learning**—Each student learns at a different pace. Personalized learning pinpoints the areas each student needs to practice, giving *every* student the support they need—when and where they need it—to be successful.
  - With **Companion Study Plan Assignments** you can assign the Study Plan as a prerequisite to a test or quiz, guiding students through the concepts they need to master.
  - With **Personalized Homework**, students take a quiz or test and receive a subsequent homework assignment that is personalized based on their performance. This way, students can focus on just the topics they have not yet mastered.

**Homework Overview** [Legend](#)  

Name Chapter 1 Homework


Last Worked 10/14/15 3:26pm

Current Score 42.86% (3 points out of 7)

This homework will **not** affect your Study Plan score.

Number of times you can complete each question: unlimited

---

 **Changes made here WILL affect your score.** [Go to Results to practice without affecting your score.](#)

▶ You received automatic credit (3 pts) for topics you mastered on Chapter 1 Skills Check Quiz.

▶ You only need to work on questions that are links below.

Questions: 7	Scored: 3	Correct: 3	Partial Credit: 0	Incorrect: 0
✓ Question 1 (1/1)	<a href="#">Question 2</a> (0/1)		<a href="#">Question 3</a> (0/1)	
✓ Question 4 (1/1)	<a href="#">Question 5</a> (0/1)		✓ Question 6 (1/1)	
<a href="#">Question 7</a> (0/1)				

# Trigonometry

This page is intentionally left blank

# 5<sup>e</sup> Trigonometry

Mark Dugopolski

Southeastern Louisiana University



**Director, Portfolio Management:** Anne Kelly  
**Courseware Portfolio Manager:** Chelsea Kharakozova  
**Courseware Portfolio Management Assistant:** Mary Catherine Connors  
**Content Producer:** Angela Montoya  
**Managing Producer:** Karen Wernholm  
**Producer:** Erin Carreiro  
**Manager, Courseware QA:** Mary Durnwald  
**Manager, Content Development:** Kristina Evans  
**Product Marketing Manager:** Stacey Sveum  
**Product Marketing Assistant:** Shannon McCormack  
**Field Marketing Manager:** Peggy Lucas  
**Field Marketing Assistant:** Adrianna Valencia  
**Senior Author Support/Technology Specialist:** Joe Vetere  
**Manager, Rights and Permissions:** Gina Cheselka  
**Text and Cover Design:** Jerilyn Bockorick  
**Full Service Vendor:** Pearson CSC  
**Full Service Project Management:** Pearson CSC, Sharon Cahill  
**Manufacturing Buyer:** Carol Melville, LSC Communications  
**Cover Image:** Pearson CSC

Copyright © 2020, 2015, 2011 by Pearson Education, Inc. All Rights Reserved. Printed in the United States of America. This publication is protected by copyright, and permission should be obtained from the publisher prior to any prohibited reproduction, storage in a retrieval system, or transmission in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise. For information regarding permissions, request forms and the appropriate contacts within the Pearson Education Global Rights & Permissions department, please visit [www.pearsoned.com/permissions/](http://www.pearsoned.com/permissions/).

PEARSON, ALWAYS LEARNING, and MYLAB are exclusive trademarks owned by Pearson Education, Inc., or its affiliates in the U.S. and/or other countries.

Unless otherwise indicated herein, any third-party trademarks that may appear in this work are the property of their respective owners and any references to third-party trademarks, logos, or other trade dress are for demonstrative or descriptive purposes only. Such references are not intended to imply any sponsorship, endorsement, authorization, or promotion of Pearson's products by the owners of such marks, or any relationship between the owner and Pearson Education, Inc., or its affiliates, authors, licensees, or distributors.

All Texas Instruments TI-84 Plus C Graphing Calculator graphs © copyright by Texas Instruments Education Technology.

#### **Library of Congress Cataloging-in-Publication Data**

Names: Dugopolski, Mark, author.  
Title: Trigonometry / Mark Dugopolski (Southeastern Louisiana University).  
Description: 5th edition. | Boston : Pearson, [2020] | Includes indexes.  
Identifiers: LCCN 2018038979 | ISBN 9780135207338 (student edition)  
Subjects: LCSH: Trigonometry—Textbooks.  
Classification: LCC QA531 .D764 2020 | DDC 516.24—dc23  
LC record available at <https://lccn.loc.gov/2018038979>



ISBN 13: 978-0-135-20733-8  
ISBN 10: 0-135-20733-9

Preface ix

Resources for Success xiii

## P

### Algebraic Prerequisites 1

**P.1** The Cartesian Coordinate System 2

**P.2** Functions 14

**P.3** Families of Functions, Transformations, and Symmetry 21

**P.4** Compositions and Inverses 31

Highlights 41

Chapter P Review Exercises 43

Chapter P Test 44

## 1

### Angles and the Trigonometric Functions 45

**1.1** Angles and Degree Measure 46

**1.2** Radian Measure, Arc Length, and Area 55

**1.3** Angular and Linear Velocity 65

**1.4** The Trigonometric Functions 73

**1.5** Right Triangle Trigonometry 83

**1.6** The Fundamental Identity and Reference Angles 99

Highlights 106

Chapter 1 Review Exercises 108

Chapter 1 Test 111

## 2

### Graphs of the Trigonometric Functions 112

**2.1** The Unit Circle and Graphing 113

**2.2** The General Sine Wave 125

**2.3** Graphs of the Secant and Cosecant Functions 137

**2.4** Graphs of the Tangent and Cotangent Functions 143

**2.5** Combining Functions 152

Highlights 159

Chapter 2 Review Exercises 160

Chapter 2 Test 162

Tying It All Together 162

## Trigonometric Identities 165

- 3.1** Basic Identities 166
- 3.2** Verifying Identities 176
- 3.3** Sum and Difference Identities for Cosine 184
- 3.4** Sum and Difference Identities for Sine and Tangent 191
- 3.5** Double-Angle and Half-Angle Identities 197
- 3.6** Product and Sum Identities 206
  - Highlights 215
  - Chapter 3 Review Exercises 217
  - Chapter 3 Test 218
  - Tying It All Together 219

## Solving Conditional Trigonometric Equations 220

- 4.1** The Inverse Trigonometric Functions 221
- 4.2** Basic Sine, Cosine, and Tangent Equations 232
- 4.3** Equations Involving Compositions 244
- 4.4** Trigonometric Equations of Quadratic Type 252
  - Highlights 260
  - Chapter 4 Review Exercises 261
  - Chapter 4 Test 263
  - Tying It All Together 264

## Applications of Trigonometry 265

- 5.1** The Law of Sines 266
- 5.2** The Law of Cosines 277
- 5.3** Area of a Triangle 287
- 5.4** Vectors 294
- 5.5** Applications of Vectors 304
  - Highlights 313
  - Chapter 5 Review Exercises 314
  - Chapter 5 Test 317
  - Tying It All Together 318





# Complex Numbers, Polar Coordinates, and Parametric Equations 319

- 6.1** Complex Numbers 320
- 6.2** Trigonometric Form of Complex Numbers 328
- 6.3** De Moivre's Theorem, Powers, and Roots 335
- 6.4** Polar Equations 341
- 6.5** Parametric Equations 352
- 6.6** Fun with Polar and Parametric Equations 358
  - Highlights 360
  - Chapter 6 Review Exercises 362
  - Chapter 6 Test 363
  - Tying It All Together 364

**Online Appendix A: Solutions to *Try This* Exercises**

**Online Appendix B: More *Outside the Box***

**Photo Credits** 365

**Answers to Exercises** A-1

**Index of Applications** I-0

**Index** I-3

**Function Gallery** I-11

This page is intentionally left blank

# PREFACE

The fifth edition of *Trigonometry* is designed for a variety of students with different mathematical needs. For those students who will take additional mathematics, this text will provide the proper foundation of skills, understanding, and insights for success in subsequent courses. For those students who will not further pursue mathematics, the extensive emphasis on applications and modeling will demonstrate the usefulness and applicability of trigonometry in the modern world. I am always on the lookout for real-life applications of mathematics, and I have included many problems that people actually encounter on the job. With an emphasis on problem solving, this text provides students an excellent opportunity to sharpen their reasoning and thinking skills. With increased problem-solving capabilities, students will have confidence to tackle problems that they encounter inside and outside the mathematics classroom.

## NEW TO THIS EDITION

- Numerous explanations, examples, and exercises have been rewritten in response to comments from users of the fourth edition.
- This edition contains many new nonstandard exercises, ranging from easy to challenging. There is nothing this author enjoys more than creating original exercises. (e.g. exercise 33, p. 274).
- Section 2.5, Combining Functions, has been rewritten to put more emphasis on producing the graphs with a graphing calculator than drawing them by hand.
- Section 3.2, Verifying Identities, has been rewritten so that there are six examples corresponding to the six-point strategy for verifying identities. The exercise set now contains four exercises corresponding to each point of the six-point strategy and they are labeled as such (e.g. exercise 15, p. 181). This arrangement will give students the opportunity to see how each point of the strategy can be used. In the mixed exercises that follow, students decide which point of the strategy will work on each exercise.
- Multiplying trigonometric binomials and factoring trigonometric expressions have been moved from Section 3.2 to Section 3.1. Now Section 3.2 concentrates solely on verifying identities using the six-point strategy.
- All graphing calculator screen shots have been redone using the TI-84 Plus CE.
- A new section has been added to Chapter 6, Fun with Polar and Parametric Equations. With polar and parametric equations, producing interesting graphs on a calculator that would be impossible to draw by hand. This section contains only one exercise on page 360. This author will award a \$100 prize to the first student who submits a correct solution. See page 360 for more details.



## CONTINUING FEATURES

With each new edition, all of the features are reviewed to make sure they are providing a positive impact on student success. The continuing features of the text are listed here.

### Strategies for Success

- **Chapter Opener** Each chapter begins with chapter opener text that discusses a real-world situation in which the mathematics of the chapter is used. Examples and exercises that relate back to the chapter opener are included in the chapter.
- **Try This** Occurring immediately after every example in the text is an exercise that is very similar to the example. These problems give students the opportunity to immediately check their understanding of the example. Solutions to all *Try*

*This* exercises are in Online Appendix A, which can be found in MyLab Math or in the Instructor Resource Center. See p. 116.

- **Summaries** of important concepts are included to help students clarify ideas that have multiple parts. See p. 235.
- **Strategies** contain general guidelines for solving certain types of problems. They are designed to help students sharpen their problem-solving skills. See p. 180.
- **Procedures** are similar to *Strategies* but are more specific and more algorithmic. *Procedures* are designed to give students a step-by-step approach for solving a specific type of problem. See p. 127.
- **Function Galleries** Located throughout the text, these function summaries are all gathered at the end of the text. These graphical summaries are designed to help students link the visual aspects of various families of functions to the properties of the functions. See p. 131.
- **Hints** Selected applications include hints that are designed to encourage students to start thinking about the problem at hand. A *Hint* logo  is used where a hint is given. See p. 218.
- **Graphing Calculator Discussions** Optional graphing calculator discussions have been included in the text. They are clearly marked by graphing calculator icons  so that they can be easily skipped if desired. Although the graphing calculator discussions are optional, all students will benefit from reading them. In this text, the graphing calculator is used as a tool to support and enhance conclusions, not to make conclusions. See p. 171.

## Section Exercises and Review

- **For Thought** Each exercise set is preceded by a set of ten true/false questions that review the basic concepts in the section, help check student understanding, and offer opportunities for writing and discussion. The answers to all *For Thought* exercises are included in the back of the *Student Edition*.
- **Exercise Sets** The exercise sets range from easy to challenging and are arranged in order of increasing difficulty. Those exercises that require a graphing calculator are optional and are marked with an icon.
- **Writing/Discussion Exercises** These exercises deepen students' understanding by giving them the opportunity to express mathematical ideas both in writing and to their classmates during small group or team discussions. See p. 243.
- **Outside the Box** Found throughout the text, these problems are designed to get students and instructors to do some mathematics just for fun. I enjoyed solving these problems and hope that you will also. The problems can be used for individual or group work. They may or may not have anything to do with the sections in which they are located and might not even require any techniques discussed in the text. So be creative and try thinking *Outside the Box*. See p. 244. The answers are given in the *Annotated Instructor's Edition* only, and complete solutions can be found in the *Instructor's Solutions Manual*. Online Appendix B contains 34 extra *Outside the Box* problems.
- **Pop Quizzes** Included at the end of every section of the text, the *Pop Quizzes* give instructors and students convenient quizzes that can be used in the classroom to check understanding of the basics. The answers appear in the *Annotated Instructor's Edition* only. New for this edition, all Pop Quiz questions are available in Learning Catalytics to assess students in real time! See p. 252.
- **Linking Concepts** This feature is located at the end of nearly every exercise set. It is a multipart exercise or exploration that can be used for individual or group work. The idea of this feature is to use concepts from the current section along with concepts from preceding sections or chapters to solve problems that illustrate the links among various ideas. Some parts of these questions are open-ended and require somewhat more thought than standard skill-building exercises. See p. 215. Answers are given in the *Annotated Instructor's Edition* only, and full solutions can be found in the *Instructor's Solutions Manual*.

## Chapter Review

- **Highlights** This end-of-chapter feature contains an overview of all of the concepts presented in the chapter, along with brief examples to illustrate the concepts.
- **Chapter Review Exercises** These exercises are designed to give students a comprehensive review of the chapter without reference to individual sections and to prepare students for a chapter test.
- **Chapter Test** The problems in the *Chapter Test* are designed to measure the student's readiness for a typical one-hour classroom test. Instructors may also use them as a model for their own end-of-chapter tests. Students should be aware that their in-class test could vary from the *Chapter Test* due to different emphasis placed on the topics by individual instructors.
- **Tying It All Together** Found at the end of most chapters in the text, these exercises help students review selected concepts from the present and prior chapters and require students to integrate multiple concepts and skills. See p. 219.
- **Index of Applications** The many applications contained within the text are listed in the *Index of Applications* that appears at the end of the text. The applications are page referenced and grouped by subject matter.

# Get the Most Out of MyLab Math

Math courses are continuously evolving to help today's students succeed. It's more challenging than ever to support students with a wide range of backgrounds, learning styles, and math anxieties. The flexibility to build a course that fits instructors' individual course formats—with a variety of content options and multimedia resources all in one place—has made MyLab Math the market-leading solution for teaching and learning mathematics since its inception.

## Preparedness

One of the biggest challenges in Trigonometry is making sure students are adequately prepared with prerequisite knowledge. For a student, having a firm grasp on foundational skills can dramatically increase success.

**MyLab Math with Integrated Review** can be used in corequisite courses or simply to help students who enter without a full understanding of prerequisite concepts. Integrated Review provides videos on review topics, Guided Visualizations illustrating concepts, and pre-made, assignable skills-check quizzes and personalized review homework assignments. Use along with our new **Integrated Review Notebook** to fully support students throughout their Trigonometry course.

12/15/18 11:59pm	◆ Chapter 2 Skills Check
12/15/18 11:59pm	▶ ● Chapter 2 Skills Review Homework
12/15/18 11:59pm	◆ Chapter 3 Skills Check
12/15/18 11:59pm	▶ ● Chapter 3 Skills Review Homework
12/15/18 11:59pm	◆ Chapter 4 Skills Check
12/15/18 11:59pm	▶ ● Chapter 4 Skills Review Homework
12/15/18 11:59pm	<ul style="list-style-type: none"> <li>You must do Chapter 4 Skills Check before starting this assignment.</li> </ul>
12/15/18 11:59pm	▶ ● Chapter 5 Skills Review Homework

# Resources for Success

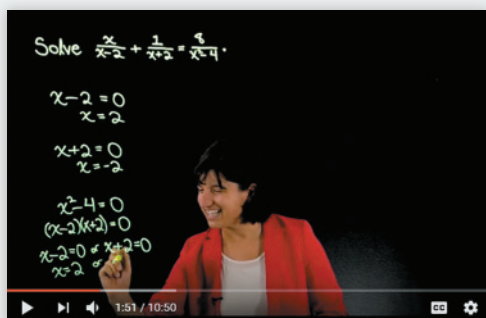
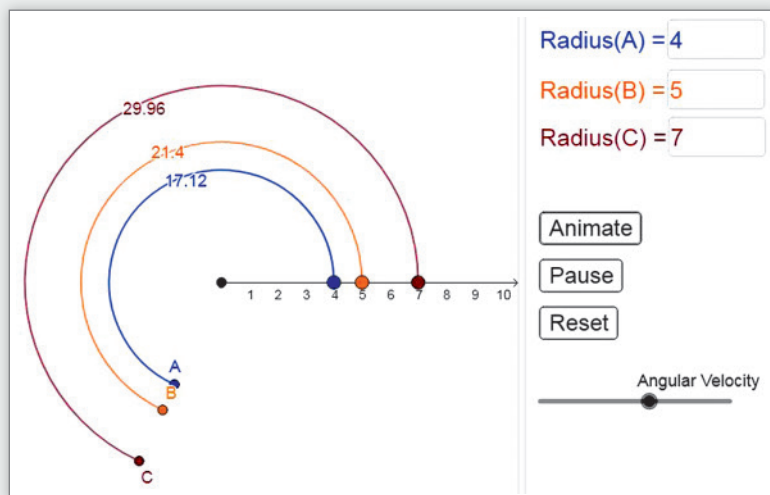
## MyLab Math with Integrated Review for *Trigonometry 5e*

by Mark Dugopolski (access code required)

MyLab Math is tightly integrated with each author's style, offering a range of book-specific multimedia resources so your students have a consistent experience across text and technology.

### New! Guided Visualizations

**Guided Visualizations** bring mathematical concepts to life, helping students visualize the concepts through directed explorations and purposeful manipulation. Guided Visualizations accompany the Integrated Review topics and key concepts from the main text, allowing students to dynamically explore and think more deeply about key ideas in Trigonometry. Guided Visualizations and accompanying assessment exercises can be assigned in MyLab Math.

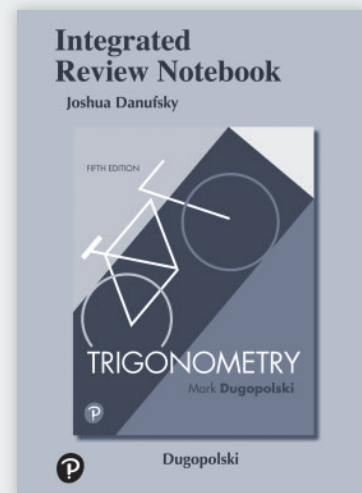


### New! Worked Example Videos

Worked example videos engage and support students outside the classroom, covering all Integrated Review prerequisite topics and key examples from the book, hand-picked by the author. Lightboard technology creates a personal experience and simulates an in-class environment. Accompanying assessment questions in MyLab Math make these brand new videos assignable.

### New! Integrated Review Notebook

This printed guide aligns to the Integrated Review topics and gives students a structured place to take notes as they work through the prerequisite objectives. Definitions, unique examples, and important concepts are highlighted. The notebook is available as a print supplement or in MyLab Math for download.



# Resources for Success

## Instructor Resources

Online resources can be downloaded from [www.pearson.com](http://www.pearson.com), or hardcopy resources can be ordered from your sales representative.

### TestGen®

TestGen ([www.pearsoned.com/testgen](http://www.pearsoned.com/testgen)) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.

### Powerpoint® Lecture Slides

(Download Only)

Classroom presentation slides are geared specifically to sequence the text. Available in MyLab Math.

### Annotated Instructor's Edition

(ISBN: 9780135207468)

This edition provides answers beside the text exercises for most exercises and in an answer section at the back of the book for all others. Groups of exercises are keyed back to corresponding examples from the section.

### Instructor's Solutions Manual

(Download Only)

This manual provides complete solutions to all text exercises, including the *For Thought* and *Linking Concepts* exercises.

### Instructor's Testing Manual

(Download only)

This resource provides six prepared test forms for each chapter of the text as well as answers. One test form is available for each section of Chapter P.

## Student Resources

Additional resources are available to help students succeed.

### Example Videos

These new videos provide comprehensive coverage of key topics in the text in an engaging format. Includes optional subtitles in English and Spanish. All videos are assignable within MyLab Math.

### Student's Solutions Manual

(ISBN: 9780135232927)

This manual provides detailed solutions to odd-numbered exercises in the text.

### Integrated Review Notebook

(ISBN: 9780135207529)

This printed guide offers a structured place to engage with the foundational concepts of Trigonometry. Each concept has a concise exposition followed by a guided example and practice exercises. Ideal for corequisite class time or independent study.

### Graphing Resources

Interactive tutorials and how-to videos are available for GeoGebra and TI-84 Plus, respectively. These resources and more can be found in the **Graphing Resources** tab in MyLab Math. Students will be able to launch GeoGebra Graphing Calculator from that tab or they can download the free app while completing their homework assignments.



# ACKNOWLEDGMENTS

Thanks to the many professors and students who have used this text in previous editions. I am always glad to hear from users of my texts. You can e-mail me at [bookinit@charter.net](mailto:bookinit@charter.net). Thanks to the following reviewers, whose comments and suggestions were invaluable in preparing this fifth edition:

Douglas B. Aichele, <i>Oklahoma State University</i>	Janet Harter, <i>Amarillo College</i>
Richard Araiza, <i>Albuquerque Technical Vocational Institute</i>	Brian Hickey, <i>East Central College</i>
Angela Beltran-Aguilar, <i>Mira Costa College</i>	Timothy D. Hudson, <i>East Carolina University</i>
Clark Bennett, <i>University of South Dakota</i>	Evan Innerst, <i>Canada College</i>
David C. Bramlett, <i>Jackson State University</i>	Christine Ingrid Keller, <i>Southwest Texas State University</i>
Thomas R. Caplinger, <i>University of Memphis</i>	Kimberly Johnson, <i>Mesa Community College</i>
Marte M. Carter, <i>Central Missouri State University</i>	Carol Miller, <i>Ivy Technical Community College–Muncie</i>
Frederick Chapple, <i>Baltimore City Community College</i>	John Nichols, <i>Oklahoma Baptist University</i>
Julane B. Crabtree, <i>Johnson County Community College</i>	Robert Parr, <i>Blinn College–Bryan</i>
Diane Dexter Ellis, <i>Mississippi State University</i>	Pat Simms, <i>Ouachita Technical College</i>
Thomas L. Fitzkee, <i>Francis Marion University</i>	Donald P. Skow, <i>University of Texas, Pan American</i>
Jeremiah A. Gilbert, <i>San Bernardino Community College</i>	Edward Swenson, <i>Forsyth Technical Community College</i>
Ken Harrelson, <i>Oklahoma City Community College</i>	Alexandra Tauson, <i>Community College of Allegheny County</i>
	Kirk C. Trigsted, <i>University of Idaho</i>
	John Vogt, <i>Newman University</i>
	Mandi Wheeler, <i>Amarillo College</i>

Thanks to Edgar N. Reyes, Southeastern Louisiana University, for working all of the exercises and writing the *Solutions Manuals*; Rebecca W. Muller, Southeastern Louisiana University, for writing the *Instructor's Testing Manual*; and Darryl Nester, Bluffton University, for writing the *Graphing Calculator Manual*; I wish to express my thanks to Deana Richmond and Christine S. Verity for accuracy checking. A special thanks to Pearson CSC, for the superb work on Production Coordination, Composition, and Illustrations.

Finally, it has been another wonderful experience working with the talented and dedicated Pearson team. I wish to thank them for their assistance, encouragement, and direction on this project: Chelsea Kharakozova, Portfolio Manager, whose ideas were central in making this revision a success; Peggy Lucas and Stacey Sveum, Marketing Managers, for creatively tying all the ideas together to get the message out; Angela Montoya, Content Producer, for keeping it all on track with poise; Jerilyn Bockorick, Text and Cover Designer, whose eye for design helped improve the book's look dramatically; Erin Carreiro, Producer, who put together a top-notch media package; Mary Catherine Connors, Portfolio Assistant, and Shannon McCormack and Adrianna Valencia, Marketing Assistants, who ran the behind-the-scenes work; and Joe Vetere, Senior Author Support/Technical Specialist, for his expertise in the production of this book.

As always, thanks to my wife, Cheryl, whose love, encouragement, understanding, support, and patience are invaluable.

**Mark Dugopolski**  
*Ponchatoula, Louisiana*

This page is intentionally left blank



# Algebraic Prerequisites

In the 1995 America's Cup race the defending *Young America* kept up a good face, but from the start it was clear that the New Zealand entry, *Black Magic*, was sailing higher and faster in the 11-knot breeze and rolling swells. For two-and-a-half years the crew had done everything they could to prepare for that moment.

Gone are the days when raw sailing ability and stamina won races like the America's Cup. New Zealand's *Black Magic* sported a mathematically created design that reduced its drag and optimized its stability and speed.

- P.1** The Cartesian Coordinate System
- P.2** Functions
- P.3** Families of Functions, Transformations, and Symmetry
- P.4** Compositions and Inverses



## WHAT YOU WILL LEARN

Preparation is just as important in trigonometry as it is in sailing. In this chapter we review some of the basics of algebra that are necessary for success in trigonometry. Throughout this chapter, you will see that even basic concepts of algebra have applications in business, science, engineering, and even sailing.

# P.1 The Cartesian Coordinate System

Table P.1

Toppings $x$	Cost $y$
0	\$ 5
1	7
2	9
3	11
4	13


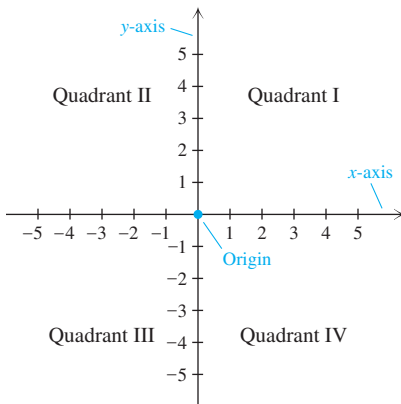



Figure P.1

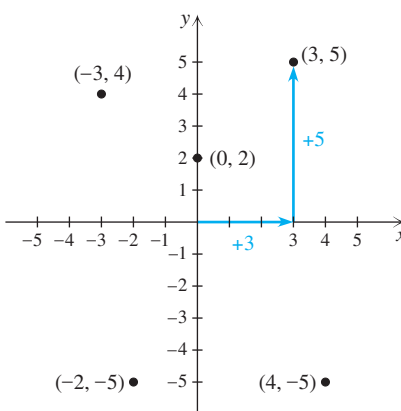


Figure P.2

In this chapter we study pairs of variables and how they are related. For example,  $p$  might represent the price of gasoline and  $n$  the number of gallons that you consume in a month;  $x$  might be the number of toppings on a medium pizza and  $y$  the cost of that pizza; or  $h$  might be the height of a two-year-old child and  $w$  the weight. To picture relationships between pairs of variables we use a two-dimensional coordinate system.

## A Two-Dimensional Coordinate System

Suppose that the cost of a medium pizza is \$5 plus \$2 per topping. Table P.1 shows a chart in which  $x$  is the number of toppings and  $y$  is the cost. To indicate that  $x = 3$  and  $y = 11$  go together, we can use the **ordered pair**  $(3, 11)$ . The ordered pair  $(2, 9)$  indicates that a 2-topping pizza costs \$9. The **first coordinate** of the ordered pair represents the number of toppings and the **second coordinate** represents the cost. The assignment of what the coordinates represent is arbitrary, but once made it is kept fixed. For example, in the present context the ordered pair  $(11, 3)$  indicates that an 11-topping pizza costs \$3, and it is not the same as  $(3, 11)$ . This is the reason they are called “ordered pairs.”

Note that parentheses are used to indicate ordered pairs and also to indicate intervals of real numbers. For example,  $(3, 7)$  could be an ordered pair or the interval of real numbers between 3 and 7. However, the meaning of this notation should always be clear from the discussion.

Ordered pairs are pictured on a plane in the **rectangular coordinate system**, or **Cartesian coordinate system**, named after the French mathematician René Descartes (1596–1650). The Cartesian coordinate system consists of two number lines drawn perpendicular to one another, intersecting at zero on each number line as in Fig. P.1. The point at which they intersect is called the **origin**. The horizontal number line is the  **$x$ -axis** and the vertical number line is the  **$y$ -axis**. The two number lines divide the plane into four regions called **quadrants**, numbered as in Fig. P.1. The quadrants do not include any points on the axes.

We call a plane with a rectangular coordinate system the **coordinate plane**, or the  **$xy$ -plane**. Every ordered pair of real numbers  $(a, b)$  corresponds to a point  $P$  in the  $xy$ -plane. For this reason, ordered pairs of numbers are often called **points**. The numbers  $a$  and  $b$  are called the **coordinates** of point  $P$ . Locating the point  $P$  that corresponds to  $(a, b)$  in the  $xy$ -plane is referred to as **plotting** or **graphing** the point, and  $P$  is called the **graph** of  $(a, b)$ .

### EXAMPLE 1 Plotting points

Plot the points  $(3, 5)$ ,  $(4, -5)$ ,  $(-3, 4)$ ,  $(-2, -5)$ , and  $(0, 2)$  in the  $xy$ -plane.

#### Solution

The point  $(3, 5)$  is located 3 units to the right of the origin and 5 units above the  $x$ -axis as shown in Fig. P.2. The point  $(4, -5)$  is located 4 units to the right of the origin and 5 units below the  $x$ -axis. The point  $(-3, 4)$  is located 3 units to the left of the origin and 4 units above the  $x$ -axis. The point  $(-2, -5)$  is located 2 units to the left of the origin and 5 units below the  $x$ -axis. The point  $(0, 2)$  is on the  $y$ -axis because its first coordinate is zero.

**TRY THIS.** Plot  $(-3, -2)$ ,  $(-1, 3)$ ,  $(5/2, 0)$ , and  $(2, -3)$ .

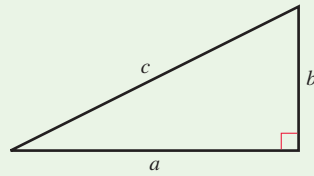
Note that for points in quadrant I, both coordinates are positive. In quadrant II the first coordinate is negative and the second is positive, whereas in quadrant III, both coordinates are negative. In quadrant IV the first coordinate is positive and the second is negative.

## The Pythagorean Theorem

To find a formula for the distance between two points in the Cartesian coordinate system we need the Pythagorean theorem. The Pythagorean theorem gives a relationship between the legs and hypotenuse of a right triangle.

### The Pythagorean Theorem

In a right triangle the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.



$$a^2 + b^2 = c^2$$

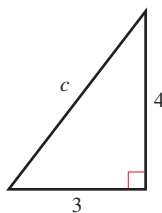
The Pythagorean theorem is probably the most widely known theorem in mathematics. It was known to the Babylonians 4000 years ago and has been proven by such notables as Euclid, Leonardo da Vinci, and James Garfield (the 20th U.S. president). You can find many proofs on the Internet. The first site that this author found contained 54 proofs.

If any two sides of a right triangle are known, the Pythagorean theorem can be used to find the third side, as shown in the next example.

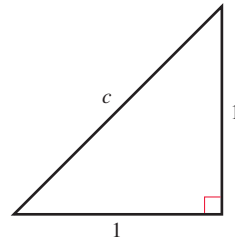
### EXAMPLE 2 Applying the Pythagorean theorem

Find the unknown side in each given triangle.

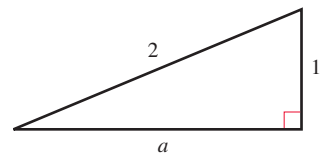
a.



b.



c.



#### Solution

a. The lengths of the legs are 3 and 4 and the hypotenuse is unknown. Use  $c^2 = a^2 + b^2$  with  $a = 3$  and  $b = 4$ :

$$c^2 = 3^2 + 4^2$$

$$c^2 = 25$$

$$c = \pm\sqrt{25} = \pm 5$$

Since the length of the hypotenuse is a positive number,  $c = 5$ .

- b. The lengths of the legs are each 1 unit and the hypotenuse is unknown. Use  $c^2 = a^2 + b^2$  with  $a = 1$  and  $b = 1$ :

$$c^2 = 1^2 + 1^2$$

$$c^2 = 2$$

$$c = \pm\sqrt{2}$$

Since the length of the hypotenuse is positive,  $c = \sqrt{2}$ .

- c. The hypotenuse is 2, one leg is 1, and the other leg is unknown. Use  $a^2 + b^2 = c^2$  with  $c = 2$  and  $b = 1$ :

$$a^2 + 1^2 = 2^2$$

$$a^2 = 3$$

$$a = \pm\sqrt{3}$$

Since the length of a leg of a triangle is positive,  $a = \sqrt{3}$ .

**TRY THIS.** The legs of a right triangle are 2 feet and 3 feet. Find the length of the hypotenuse.

The Pythagorean theorem is still used by builders in squaring up foundations. They know that 3-4-5, 6-8-10, 9-12-15, and so on, are all right triangles. For example, a builder will measure 9 feet and 12 feet (the legs) on two boards that are supposed to meet at a right angle. If the diagonal (or hypotenuse) measures 15 feet, the builder can be certain that the boards meet at a right angle. Builders also use construction calculators that can determine the length of the hypotenuse when the lengths of the two legs are entered.

## Simplifying Square Roots

The exact lengths of the missing sides of the triangles in Example 2 are  $\sqrt{2}$  and  $\sqrt{3}$ . Values for these roots, such as 1.414 and 1.732, that you get from a calculator are approximate. Since these numbers are irrational, they can't be expressed exactly in decimal form. In trigonometry we will often encounter square roots in their exact form. There are two rules to remember when working with square roots:

### Rule: Product and Quotient

For any real numbers  $a$  and  $b$

1.  $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$  Product rule for square roots

2.  $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$  ( $b \neq 0$ ) Quotient rule for square roots

provided that all of the roots are real numbers.

When using square roots to express exact results we use the product and quotient rules to write them in *simplified form*. (Note: In  $\sqrt{a}$ , the **radicand** is  $a$ .)

### Definition: Simplified Form for Square Roots

An expression involving a square root is in **simplified form** if it has

1. *no* perfect squares as factors of the radicand,
2. *no* fractions in the radicand, and
3. *no* square roots in a denominator.

Removing a square root from a denominator is called **rationalizing the denominator**. The next example shows how the product and quotient rules are used to rationalize denominators and get radical expressions into simplified form.

### EXAMPLE 3 Simplifying square roots

Write each expression in simplified form.

a.  $\sqrt{20}$    b.  $\sqrt{\frac{5}{9}}$    c.  $\sqrt{\frac{32}{5}}$

#### Solution

a.  $\sqrt{20} = \sqrt{4 \cdot 5}$    Identify 4 as a perfect square factor.  
 $= \sqrt{4} \cdot \sqrt{5}$    Product rule for square roots  
 $= 2\sqrt{5}$    Simplify  $\sqrt{4}$ .

b.  $\sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}}$    Quotient rule for square roots  
 $= \frac{\sqrt{5}}{3}$    Simplify  $\sqrt{9}$ .

c.  $\sqrt{\frac{32}{5}} = \frac{\sqrt{32}}{\sqrt{5}}$    Quotient rule for square roots  
 $= \frac{\sqrt{16} \cdot \sqrt{2}}{\sqrt{5}}$    Product rule for square roots  
 $= \frac{4 \cdot \sqrt{2}}{\sqrt{5}}$    Simplify.  
 $= \frac{4 \cdot \sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}}$    Rationalize the denominator.  
 $= \frac{4\sqrt{10}}{5}$    Simplified form

**TRY THIS.** Simplify  $\sqrt{50}$ ,  $\sqrt{\frac{3}{16}}$ , and  $\sqrt{\frac{12}{7}}$ .

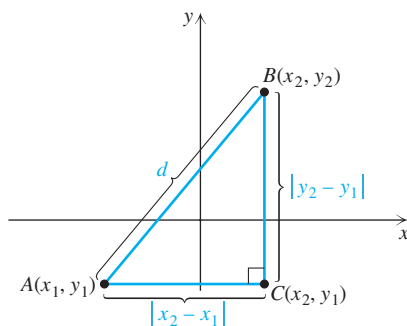


Figure P.3

### The Distance Formula

Consider the points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  shown in Fig. P.3. Let  $d$  represent the length of the line segment  $\overline{AB}$ . Now  $\overline{AB}$  is the hypotenuse of the right triangle in Fig. P.3. The distance between  $A$  and  $C$  is  $|x_2 - x_1|$  and the distance between  $B$  and  $C$  is  $|y_2 - y_1|$ . Using the Pythagorean theorem we have

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2.$$

The absolute value symbols can be replaced by parentheses, because  $|a - b|^2 = (a - b)^2$  for any real numbers  $a$  and  $b$ . Since the distance between two points is nonnegative, we have the following formula.

### The Distance Formula

The distance  $d$  between the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

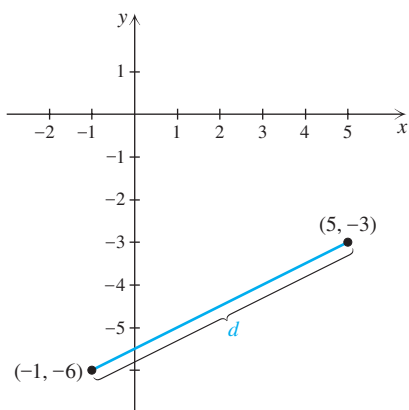


Figure P4

**EXAMPLE 4** Applying the distance formula

Find the distance between  $(5, -3)$  and  $(-1, -6)$ .

**Solution**

Let  $(x_1, y_1) = (5, -3)$  and  $(x_2, y_2) = (-1, -6)$ . See Fig. P.4. The distance is the same regardless of which point is chosen as  $(x_1, y_1)$ , or  $(x_2, y_2)$ . Substitute these values into the distance formula:

$$\begin{aligned} d &= \sqrt{(-1 - 5)^2 + (-6 - (-3))^2} \\ &= \sqrt{(-6)^2 + (-3)^2} \\ &= \sqrt{36 + 9} = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5} \end{aligned}$$

The exact distance between the points is  $3\sqrt{5}$ . Note that  $\sqrt{36 + 9} \neq \sqrt{36} + \sqrt{9}$ , but  $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5}$ .

**TRY THIS.** Find the distance between  $(-3, -2)$  and  $(-1, 4)$ .

**The Midpoint Formula**

When you average two test scores (by finding their sum and dividing by 2) you are finding a number midway between the two scores. Likewise, the midpoint of the line segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$  is found by dividing the sum of the  $x$ -coordinates by 2 and the sum of the  $y$ -coordinates by 2. See Fig. P.5.

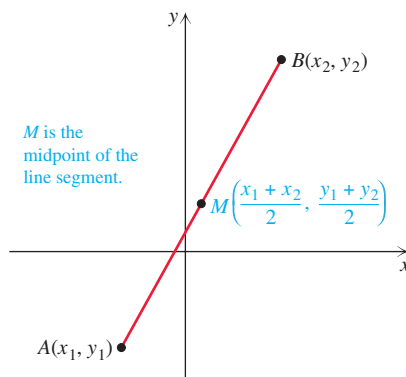


Figure P5

**The Midpoint Formula**

The midpoint of the line segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

The midpoint formula is illustrated in the next example. Note the strange-looking coordinates in part (b). In trigonometry we will often perform arithmetic with the number  $\pi$ .

**EXAMPLE 5** Applying the midpoint formula

Find the midpoint of the line segment with the given endpoints.

- a.  $(1, -3), (5, 4)$     b.  $\left(\frac{\pi}{2}, 0\right), (\pi, 0)$



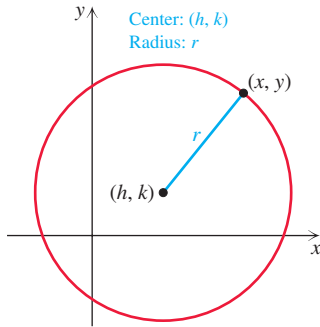


Figure P.6

**Theorem: Equation for a Circle**

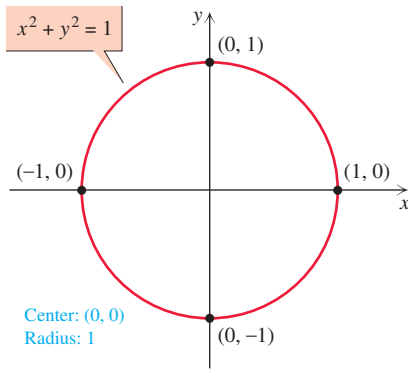


Figure P.7

**Solution**

a. To find the midpoint add the coordinates and divide by 2:

$$\left( \frac{1 + 5}{2}, \frac{-3 + 4}{2} \right) = \left( 3, \frac{1}{2} \right)$$

b. To find the midpoint add the coordinates and divide by 2:

$$\left( \frac{\frac{\pi}{2} + \pi}{2}, \frac{0 + 0}{2} \right) = \left( \frac{\frac{\pi}{2} + \frac{2\pi}{2}}{2}, 0 \right) = \left( \frac{\frac{3\pi}{2}}{2}, 0 \right) = \left( \frac{3\pi}{4}, 0 \right)$$

**TRY THIS.** Find the midpoint of the line segment with endpoints (3, -2) and (5, 6).

**The Circle**

A **circle** is the set of all points in a plane that lie a fixed distance from a given point in the plane. The fixed distance is the **radius** and the given point is the **center**. The distance formula can be used to write an equation for a circle with center (h, k) and radius r (r > 0). A point (x, y) is on the circle shown in Fig. P.6 if and only if it satisfies the equation

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

Using the definition of square root, we can write the following equation.

The equation for a circle with center (h, k) and radius r for r > 0 is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The form  $(x - h)^2 + (y - k)^2 = r^2$  is the **standard form** for the equation of a circle. If (h, k) is (0, 0), then the circle is centered at the origin and its equation is of the form  $x^2 + y^2 = r^2$ . If r = 1, the circle is called a **unit circle**.

**EXAMPLE 6 Graphing a circle**

Sketch the graph of each circle.

- a.  $x^2 + y^2 = 1$     b.  $(x - 1)^2 + (y + 2)^2 = 9$

**Solution**

a. The circle has center (0, 0) and radius 1. You can draw the circle as shown in Fig. P.7 with a compass. To draw the circle by hand locate the points that lie 1 unit above, below, right, and left of the center as shown in Fig. P.7. Then sketch a circle through these points.

b. This circle has center (1, -2) and radius 3. Plot the center and the points that lie 3 units above, below, left, and right of the center as shown in Fig. P.8. Then sketch a circle through these points.

To support the conclusions of part (b) with a graphing calculator you must first solve the equation for y:

$$\begin{aligned} (x - 1)^2 + (y + 2)^2 &= 9 \\ (y + 2)^2 &= 9 - (x - 1)^2 \\ y + 2 &= \pm \sqrt{9 - (x - 1)^2} \\ y &= -2 \pm \sqrt{9 - (x - 1)^2} \end{aligned}$$

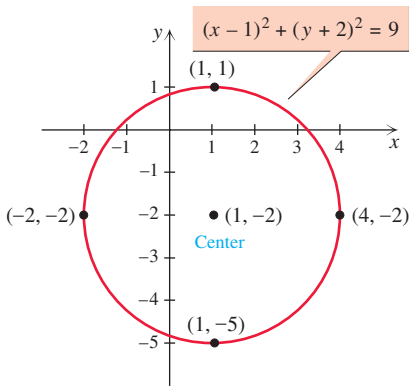
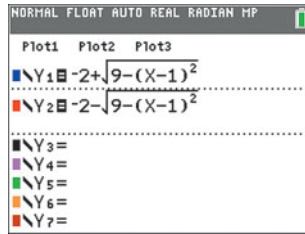
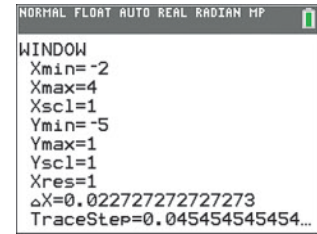


Figure P8

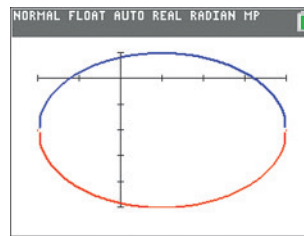
Now enter  $y_1$  and  $y_2$  as shown in Fig. P.9(a). Set the viewing window as shown in Fig. P.9(b). The graph in Fig. P.9(c) supports our previous conclusions. A circle will look round only if the same unit distance is used on both axes.



(a)



(b)



(c)

Figure P9

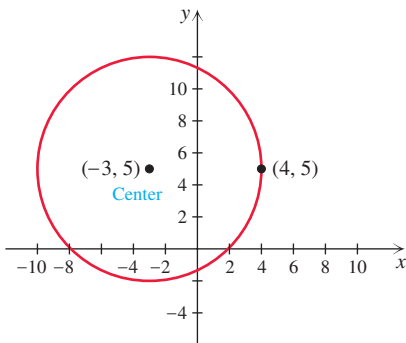


Figure P10

**TRY THIS.** Sketch the graph of  $(x + 2)^2 + (y - 4)^2 = 25$ .

In the next example we write the equation for a circle from a description of the circle.

### EXAMPLE 7 Writing the equation of a circle

Write the standard equation for a circle with center  $(-3, 5)$  and passing through  $(4, 5)$  as shown in Fig. P.10.

#### Solution

Since the distance between the center  $(-3, 5)$  and  $(4, 5)$  is 7 units, the radius of the circle is 7. Use  $h = -3$ ,  $k = 5$ , and  $r = 7$  in the standard equation for a circle:

$$(x - (-3))^2 + (y - 5)^2 = 7^2$$

So the equation of the circle is  $(x + 3)^2 + (y - 5)^2 = 49$ .

**TRY THIS.** Write the standard equation for the circle with center  $(2, -1)$  and passing through  $(3, 6)$ .

## The Line

Any circle in the coordinate plane is the solution set to an equation of the form  $(x - h)^2 + (y - k)^2 = r^2$ . Likewise, any straight line in the coordinate plane is the solution set to an equation of another form.

**Definition: Linear Equation in Two Variables (Standard Form)**

A **linear equation** in two variables  $x$  and  $y$  is an equation of the form

$$Ax + By = C$$

where  $A$ ,  $B$ , and  $C$  are real numbers and  $A$  and  $B$  are not both zero.

The graph of any linear equation is a straight line. A linear equation can be written in many different forms. The equations

$$x = 5 - y, \quad y = \frac{1}{2}x - 9, \quad x = 4, \quad \text{and} \quad y = 5$$

are all linear equations because they could all be written in the form  $Ax + By = C$ .

There is only one line containing any two distinct points in the  $xy$ -plane. So to graph a linear equation we need locate only two points that satisfy the equation and draw a line through them. But which points do we use? Since all lines look alike, what distinguishes one line from another is its location. The best way to show the location is to show the points where the line crosses the  $x$ - and  $y$ -axes. These points are called the  **$x$ -intercept** and  **$y$ -intercept**.

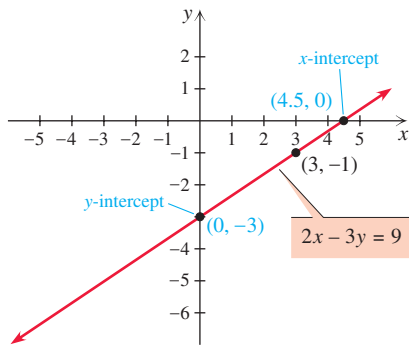


Figure P.11

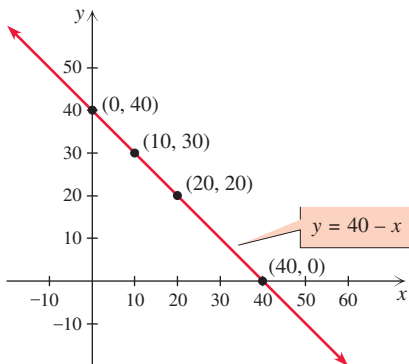


Figure P.12

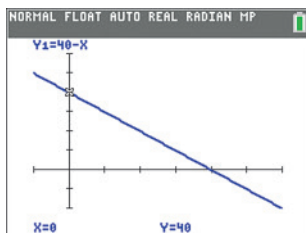


Figure P.13

**EXAMPLE 8** Graphing lines and showing the intercepts

Graph each equation. Find and show the intercepts.

- a.  $2x - 3y = 9$     b.  $y = 40 - x$

**Solution**

a. Since the  $y$ -coordinate of the  $x$ -intercept is 0, we replace  $y$  with 0 in the equation:


$$\begin{aligned} 2x - 3(0) &= 9 \\ 2x &= 9 \\ x &= \frac{9}{2} \end{aligned}$$

To find the  $y$ -intercept, we replace  $x$  with 0 in the equation:

$$\begin{aligned} 2(0) - 3y &= 9 \\ -3y &= 9 \\ y &= -3 \end{aligned}$$

The  $x$ -intercept is  $(9/2, 0)$  and the  $y$ -intercept is  $(0, -3)$ . Locate the intercepts and draw the line as shown in Fig. P.11. To check, locate a point such as  $(3, -1)$ , which also satisfies the equation, and see if the line goes through it.

- b. If  $x = 0$ , then  $y = 40 - 0 = 40$  and the  $y$ -intercept is  $(0, 40)$ . If  $y = 0$ , then  $0 = 40 - x$  or  $x = 40$ . The  $x$ -intercept is  $(40, 0)$ . Draw a line through these points as shown in Fig. P.12. Check that  $(10, 30)$  and  $(20, 20)$  also satisfy  $y = 40 - x$  and the line goes through these points.

 The calculator graph shown in Fig. P.13 is consistent with the graph in Fig. P.12. Note that the viewing window is set to show the intercepts.

**TRY THIS.** Graph  $2x + 5y = 10$  and determine the intercepts.